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Supression and creation of chaos in a periodically forced Lorenz system

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Periodic forcing is introduced into the Lorenz model to study the effects of time-dependent forcing on the behavior of the system. Such a nonautonomous system stays dissipative and has a bounded attracting set which all trajectories finally enter. The possible kinds of attracting sets are restricted to periodic orbits and strange attractors. A large-scale survey of parameter space shows that periodic forcing has mainly three effects in the Lorenz system depending on the forcing frequency: (i) Fixed points are replaced by oscillations around them; (ii) resonant periodic orbits are created both in the stable and the chaotic region; (iii) chaos is created in the stable region near the resonance frequency and in periodic windows. A comparison to other studies shows that part of this behavior has been observed in simulations of higher truncations and real world experiments. Since very small modulations can already have a considerable effect, this suggests that periodic processes such as annual or diurnal cycles should not be omitted even in simple climate models.

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I. INTRODUCTION

Many nonlinear dynamical systems have aperiodic solutions that show an extreme sensitivity to changes in initial conditions so that two neighboring trajectories diverge exponentially. These nonlinear models seem to reproduce the behavior of a large number of physical systems. The necessarily limited knowledge of initial conditions prevents an accurate prediction of the future behavior of these systems. In recent years much attention has been paid to methods and processes that might be able to suppress chaos or even control it (see Ref. [1] for an overview over the literature until 1992). For example, the addition of external periodic forcing has been applied in several nonlinear systems, such as the driven pendulum [2], the perturbed sine-Gordon equations [3,4], and the Bonhoeffer–Van der Pol (BVP) oscillator [1].

Of special—not only scientific—interest in this context is the climate system and its future behavior [5]. Motion in the climate system has been known as chaotic [7]. Yet there is considerable evidence that there are strong regular signals in climatic records [8]. The regular time varying external forcing of the climate system seems to induce both chaotic and regular behavior under certain circumstances. To understand more of the processes that might be involved, we use the so-called quasi-geostrophic model of atmospheric circulation together with periodic forcing to study the response of the system to periodic signals. This corresponds to the control method mentioned above, but the motivation is very different.

The model was introduced by Lorenz [6] as

$$\begin{aligned} \dot{y}_1 &= A_1 y_2 y_3 - \nu_0 y_1 + A_2 F, \\ \dot{y}_2 &= -A_1 y_1 y_3 - \nu_0 y_2 - A_3 y_3, \\ \dot{y}_3 &= -a\nu_0 y_3 + A_4 y_2, \end{aligned} \tag{1}$$

where all coefficients are constants and F represents the radiative heating of the atmosphere (a detailed description of the model is given in [6]). For constant forcing, these equations can be linearly transformed [6] into the original Lorenz equations [7]

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= xy - bz, \end{aligned} \tag{2}$$

with (positive) parameters σ, r, and b.

In this study, we will add a periodic forcing function to the constant forcing term F (ω is the number of periods per day)

$$F(t) = F_1 + A_5 \sin(\omega t), \tag{3}$$

which leaves the Z_2 symmetry of the system unchanged. The periodic term can be regarded as a representation of annual, daily, or other periodic processes in the thermal forcing of the atmosphere.

Various methods of controlling chaos have been applied in the Lorenz system: Periodic variation of the Rayleigh number [9], an amplitude-dependent Rayleigh number [10], continuous feedback control [11], and stochastic control [12]. Periodic variation of the Rayleigh number has been studied experimentally for Rayleigh-Bénard convection [13–15]. The addition of an external forcing function is not only unique to simulate the real forcing of the atmosphere mathematically, but also has advantages compared to the other methods, for it allows us to deduce some of its properties analytically and enables the application of the procedure of Shimada and Nagashima to calculate Lyapunov exponents [16].

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We will start, accordingly, with a global analysis, which will show that the periodically forced Lorenz system stays bounded and dissipative, while the only observable attracting sets are strange attractors and periodic orbits. Section III describes the numerical method for the calculation of the Lyapunov dimension. The results will be presented in Sec. IV. We will see that periodic forcing is able both to create and to suppress chaos by resonant interaction. In Sec. V we will relate our results to studies of higher-order truncations [17] and experimental studies. Finally, Sec. VI contains a brief summary and our conclusions.

II. GLOBAL PROPERTIES

Because of the time-dependent forcing function $F(t)$ system (1) belongs to an entirely different class from the original Lorenz system (2): the nonautonomous systems. The well-known properties of the Lorenz system such as boundedness of the trajectories or existence of attracting sets [18] are therefore no more guaranteed. Before we study the effects of periodic forcing we first examine which of these properties are preserved by the nonautonomous system and which changes have to be considered.

A. Boundedness of the trajectories

Since we have a periodically forced system, we might expect some resonance phenomena or even a resonance catastrophe. This possibility is excluded by the boundedness theorem we will prove in this section. Boundedness of the trajectories is an essential requirement for a reasonably realistic model of the Earth's atmosphere, where most of the dependent variables are bounded.

We consider the general case of a dynamical system having the form

$$\dot{w}_i = \sum_{j,k} a_{ijk} w_j w_k - \sum_j b_{ij} w_j + F_i(t), \qquad (4)$$

where the coefficients a_{ijk} and b_{ij} are constant in time and the functions F_i have a global maximum. Analogously to [6] we define a quadratic form $B = \sum_{i,j} b_{ij} w_i w_j$, and a trilinear form $A = \sum_{i,j,k} a_{ijk} w_i w_j w_k$. The square distance in phase space between a trajectory and the origin is $R^2 = \sum_i w_i^2$. We obtain

$$\dot{R} = \frac{A - B + \sum_i F_i(t) w_i}{R}. \qquad (5)$$

Let $A_{\max}$ denote the maximum of A on the unit sphere $R = 1$, and $B_{\min}$ the minimum of B on the unit sphere. Because of the boundedness of the functions $F_i(t)$ we know that a maximum $F_{\max}$ of $\sum_i F_i(t) w_i$ exists on the unit sphere. It follows that

$$\dot{R} \leq A_{\max} R^2 - B_{\min} R + F_{\max}. \qquad (6)$$

In the case of system (1) we see that A is always zero because $a_{123} = A_1 = -a_{231}$ and all other $a_{ijk} = 0$. For B to be positive definite all eigenvalues of the symmetric matrix $\frac{1}{2}(b_{ij} + b_{ji})$ must be positive, which is the case if $\nu_0 \geq 0$ and

$$4a\nu_0 \geq (A_3 - A_4)^2. \qquad (7)$$

Since $F_1 + A_5 \sin(\omega t)$ is a bounded function we can find a $F_{\max} = A_2(F_1 + A_5)$. It follows that $\dot{R} < 0$ if R exceeds a certain value R_0, so that all orbits ultimately enter and remain in a bounded ellipsoid $\mathcal{E} = R_0 + \epsilon$, where ϵ can be any small positive quantity.

B. Possible attracting sets

The properties we have found so far already restrict the type of attracting invariant sets under the flow $\mathbf{F}$ we expect to observe in this system: (1) It is obvious that in the dissipative, nonautonomous system (1) neither stationary points, nor completely unstable periodic orbits can be found. (2) If we write system (1) in the form $\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}, t)$, the divergence of the flow is constant and stays the same over the whole phase space:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial y_1} + \frac{\partial F_2}{\partial y_2} + \frac{\partial F_3}{\partial y_3} = -\nu_0(2 + a) < 0. \qquad (8)$$

If we choose an arbitrary, but fixed, volume V in phase space we can integrate the divergence over V:

$$\int_V \nabla \cdot \mathbf{F} \, dV = -\nu_0(2 + a)V. \qquad (9)$$

It follows that the flow through its surface S must be negative (for $\nu_0, a > 0$):

$$\int_S \mathbf{F} \cdot d\mathbf{o} = \int_V \nabla \cdot \mathbf{F} \, dV < 0. \qquad (10)$$

If a fixed volume in phase space is an attractor of system (1) then the flow has to be tangential to its surface for $t \to \infty$. The scalar product $\mathbf{F} \cdot d\mathbf{o}$ would be always 0 and, therefore, $\int_S \mathbf{F} \cdot d\mathbf{o} = 0$, which contradicts (10). Consequently, invariant tori and quasiperiodic behavior cannot occur.

We already see that in the generic situation of a numerical survey we are left only with two kinds of observable attracting sets: periodic orbits and strange attractors.

III. NUMERICAL CALCULATION OF LYAPUNOV DIMENSION

A. Choice of the parameters

The parameter values are chosen to be the same as in [6]. Their numerical values are, accordingly, $A_1 = \frac{8}{9}\sqrt{3}$, $A_2 = \frac{1}{9}$, $A_3 = -\frac{\sqrt{3}}{18}$, $A_4 = -\frac{\sqrt{3}}{50}$, $a = 3$, and $\nu_0 = \frac{1}{48}$. These values fulfill condition (7). We will limit the parameter range of F_1 of our numerical investigation to the interval $[0, 0.5]$. A_5 will be varied between 0 and F_1 so that F is always larger than zero. The period of the forcing will be between 2 and 1000 days (corresponding to $\omega = 0.5$ and $\omega = 0.001$ periods per day). The reasons for this choice will become clearer in the following sections. The unit of time for the integration is chosen to be $\Delta t = 10\,800$ s=3 h. We used the fourth-order Taylor-series scheme of Lorenz [6] to integrate the model, which consists of evaluating the first four derivatives of y_1, y_2, and y_3 at time t and then letting

$$y_i(t+\Delta t) = \sum_{k=0}^{4} \left[\frac{d^k}{dt^k} y_i(t)\right] \frac{\Delta t^k}{k!}. \quad (11)$$

B. Lyapunov dimension

There are several methods to characterize the behavior of dynamical systems such as Fourier analysis, or the first return map of subsequent maxima [7]. In our system we have three control parameters F_1, A_5, and ω, which makes it more convenient to use a generalized dimension measure such as the Hausdorff-Besikovich dimension or the correlation dimension [19]. In this study we will use the Lyapunov dimension to characterize the general behavior of system 1, which is given by the Kaplan-Yorke conjecture

$$D_L \equiv j + \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_j}{|\lambda_{j+1}|}, \quad (12)$$

where the λ_i are the Lyapunov exponents ordered in descending order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ and j is the largest integer for which $\lambda_1 + \lambda_2 + \cdots + \lambda_j \geq 0$.

The Lyapunov exponents describe the expansion rate of a small deviation in the initial conditions [20]. In a three-dimensional system three directions are possible. If the trajectory moves on a strange attractor, at least one Lyapunov exponent is positive, since chaotic motion is characterized by high sensitivity to small changes in initial conditions. We will use the numerical procedure of Shimada and Nagashima [16] to calculate the Lyapunov exponents. The derivation of this procedure is only given for autonomous systems, but it is also valid in nonautonomous systems since it uses only general properties of linear equations.

The Kaplan-Yorke formula (12), however, does not hold for all cases. There are several counterexamples [21], but in the Lorenz system under constant forcing this relation is valid. This makes the application of the Lyapunov dimension interesting for our study: it does not need large computational efforts and can be used conveniently to map the behavior of the system. Results show that periodic orbits could be distinguished from chaotic orbits by a considerable difference in the numerical value of the Lyapunov dimension.

IV. THE EFFECTS OF PERIODIC FORCING

A. Parameter domains under constant forcing

For our choice of parameters, the parameter range of F can be divided into three domains according to the characteristic qualtitative behavior of the quasigeostrophic model (1) under constant forcing [18]: (1) Domain I: For $F \in [0, 0.0163]$ there is one globally stable stationary point. (2) Domain II: In the interval $[0.0163, 0.10785]$ stability is transferred to a symmetric pair of stationary points. The linearized flow near these points has three real negative eigenvalues only when F is very close to 0.0163, otherwise we have one real root and a pair of complex conjugate eigenvalues with negative real parts. (3) Domain III: For $F > 0.10785$ no stable stationary points exist, the trajectory moves in most cases aperiodically on a strange attractor, except in the periodic windows where periodic orbits are stable [18].

We will see that this partition is also useful for the periodically forced system, because the system's response will be qualitatively different in each domain.

B. Lyapunov dimension maps

To get an overview over the effects of periodic forcing we mapped Lyapunov dimension D_L for several fixed values of constant forcing F_1 varying A_5 and ω in the specified ranges. As expected periodic orbits showed up in areas with D_L near 1, while chaos was detected in areas of higher D_L. In each domain we obtained a qualitatively different map: (1) In domain I only a flat plane of $D_L = 1$ was visible, which leads to the conclusion that only periodic orbits exist in this domain. (2) The map of domain II (Fig. 1) shows high D_L near the complex eigenvalues of the system, where we find chaos or at least transient chaos. In the rest of the parameter range there is again a plane of $D_L = 1$ associated with periodic orbits. (3) In domain III (Fig. 2) we observe D_L near 2 indicating the existence of a strange attractor, interrupted by a "valley" near the complex eigenvalues of the system and some isolated "holes" where periodic orbits exist.

These maps allow a classification of the wide variety of phenomena created by periodic forcing in the quasi-

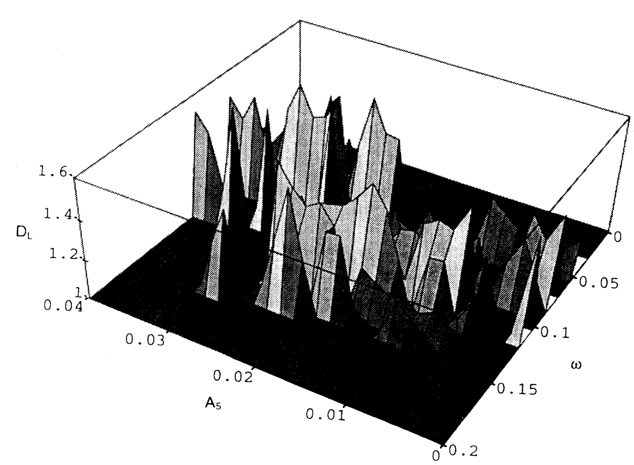

FIG. 1. Lyapunov dimension D_L for $F_1 = 0.04$ (domain II), $A_5 \in [0, 0.04]$ and $\omega \in [0, 0.2]$ (oscillations per day).

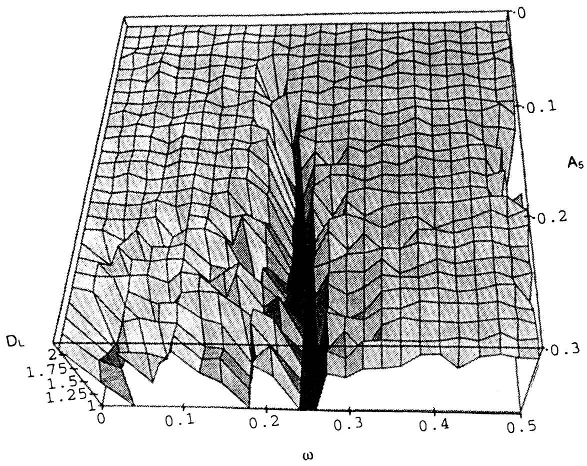

FIG. 2. Lyapunov dimension D_L for $F_1 = 0.3$ (domain III), $A_5 \in [0, 0.3]$ and $\omega \in [0, 0.5]$ (oscillations per day).

geostrophic model. There are two basic types of behavior depending on the amplitude A_5 of periodic forcing. The first type occurs when A_5 is relatively low, so that the total forcing $F(t)$ never leaves one domain, while the second type is observed when A_5 is high enough to produce overlapping effects between different domains.

C. Effects without overlapping

The Lyapunov dimension map for maximal periodic forcing ($A_5 = F_1$) is shown in Fig. 3. It is seen that the forcing frequency is the significant parameter that determines the response of the system, since all phenomena occur near the eigenfrequency and its multiples. The effects are qualitatively different in each domain: (1) Throughout domain I we observe a simple oscillation around the stationary point of the system under constant forcing. For very low frequencies the system follows quasistatically the $F(t)$-dependent stationary point (Fig. 4). (2)

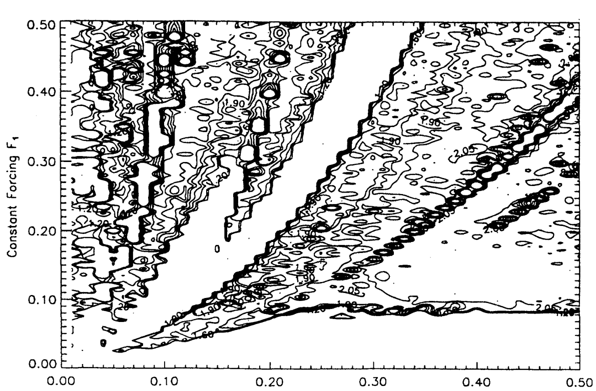

FIG. 3. Contour map of Lyapunov dimension for maximal periodic forcing ($A_5 = F_1$), $F_1 \in [0, 0.5]$ and $\omega \in [0, 0.5]$ (oscillations per day).

Similar effects occur in domain II for frequencies sufficiently far from the complex eigenvalues, e.g., oscillations around the stationary points for higher frequencies (Fig. 5), and a quasistatic orbit for very low forcing frequencies (Fig. 6). Only near the eigenfrequency of the system do we find a new phenomenon: the creation of chaos, transient chaos (Fig. 7), and complicated subharmonic periodic orbits (Fig. 8). (3) In domain III the chaotic behavior remains essentially unchanged, except near the eigenfrequency associated with the nonstable stationary points. Here the strange attractor is replaced by a periodic orbit (Fig. 9). For certain combinations of A_5, F_1, and ω there exist also isolated periodic orbits. (4) In the periodic windows the periodic orbits are almost immediately destroyed if the forcing frequency is not equal to their own frequency. On the other hand, these orbits are stabilized beyond their usual range of existence, if the forcing is appropriately adjusted.

D. Overlapping effects

There are two types of effects if the forcing $F(t)$ wanders between different domains: (1) The domain boundaries are shifted due to different convergence times, so that the oscillations of domain I are able to exist also in the marginal region of domain II (Fig. 4). For very low forcing frequencies we find that the periodic orbits of domain II can be destabilized if $F(t)$ spends sufficient time in the chaotic domain III. (2) For low forcing frequencies we observe mixed orbits that show periodic change between the quasistatic orbit of domain II and chaos (Fig. 10) or between the quasistatic orbits of domains I and II (Fig. 11).

E. Resonance

The previously described results imply an involvement of resonance in the explanation of the occurring phenomena. Qualitative changes of the system's behavior occur only for frequencies near the eigenfrequency of the system with constant forcing. The effect is largely independent of the amplitude of periodic forcing. In parameter ranges with other frequencies the underlying structure of the system under constant forcing is not essentially changed, e.g., stationary points are replaced by small oscillations around them, and chaos remains unchanged. So creation of chaos in the stable domains I and II and of periodic orbits in the chaotic domain take place only in a small frequency range around the complex eigenvalues.

Three reasons support this assumption: (1) The amplitude of the oscillations in domain II increases continuously when the frequency of the periodic forcing is

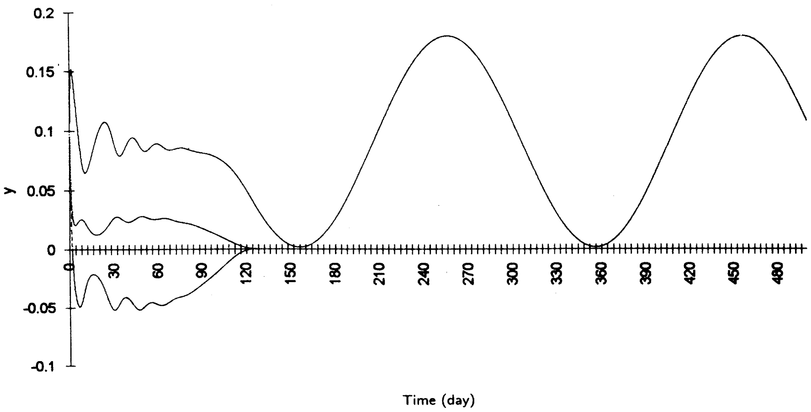

FIG. 4. The quasistatic orbit of domain I dominates over the periodic orbit of domain II near the domain boundary at $F_1 = 0.016$, $A_5 = 0.016$, and $\omega = 0.005$ oscillations per day.

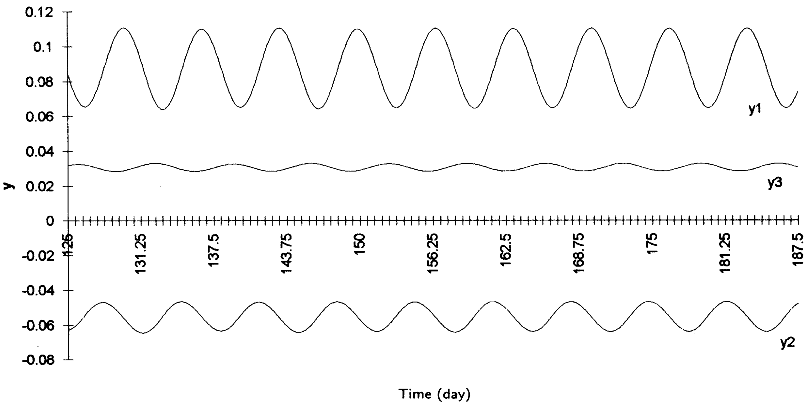

FIG. 5. Oscillation around a stationary point in domain II at $F_1 = 0.04$, $A_5 = 0.02$, and $\omega = 0.15$ oscillations per day.

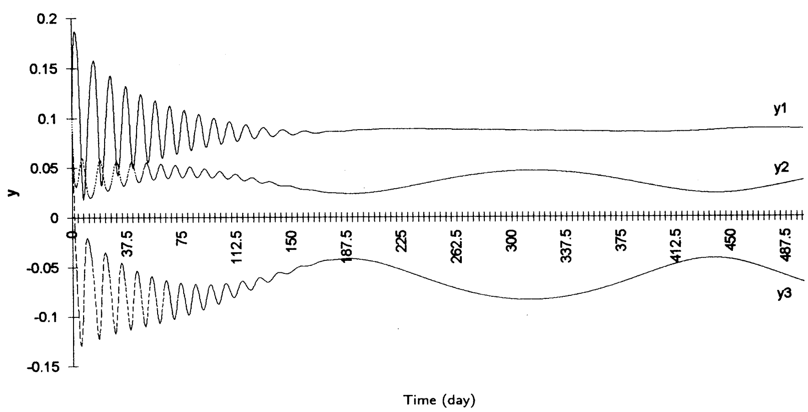

FIG. 6. A quasistatic orbit following a $F(t)$-dependent stationary point of domain II at $F_1 = 0.05$, $A_5 = 0.02$, and $\omega = 0.004$ oscillations per day.

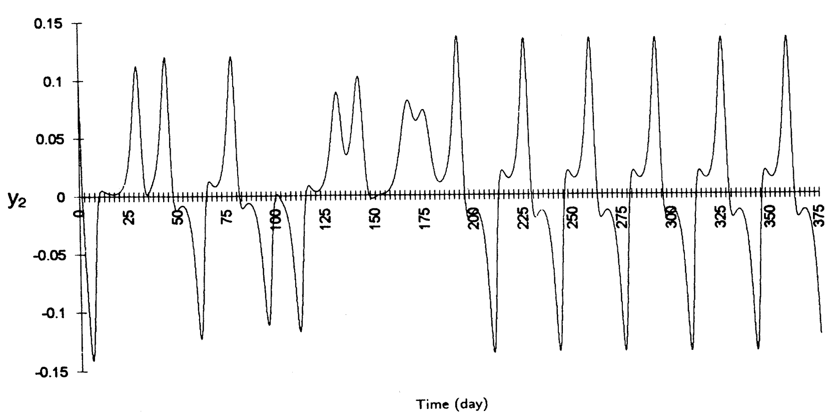

FIG. 7. Transient chaos preceding a resonant periodic orbit at $F_1 = 0.04$ (domain II), $A_5 = 0.02$ and $\omega = 0.06$ oscillations per day.

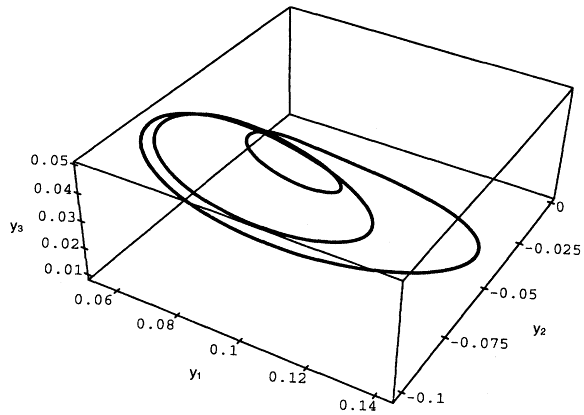

FIG. 8. Phase portrait of a resonant periodic orbit in domain II at $F_1 = 0.04$, $A_5 = 0.02$, and $\omega = 0.02$ oscillations per day.

decreased towards the eigenfrequency. (2) In domain I where no complex eigenvalues exist we observe neither chaotic behavior nor new periodic orbits. (3) Periodic forcing is able to stabilize periodic orbits in parameter ranges where they are unstable under constant forcing if the forcing frequency is equal to their own frequency.

V. HIGHER-ORDER MODELS AND EXPERIMENTS

How far are these results extendible to real world problems? The main problem is the justification of the extreme truncation of system (1) or (2). If a center manifold reduction is performed for the equations of Rayleigh-Bénard convection, one arrives at three essential modes which govern the overall behavior of the system. If the solution of the partial differential equation is expanded into these three modes and the other modes are discarded one arrives after rescaling at the Lorenz equations (2). Therefore, the use of a three-mode truncation appears appropriate from a mathematical point of view.

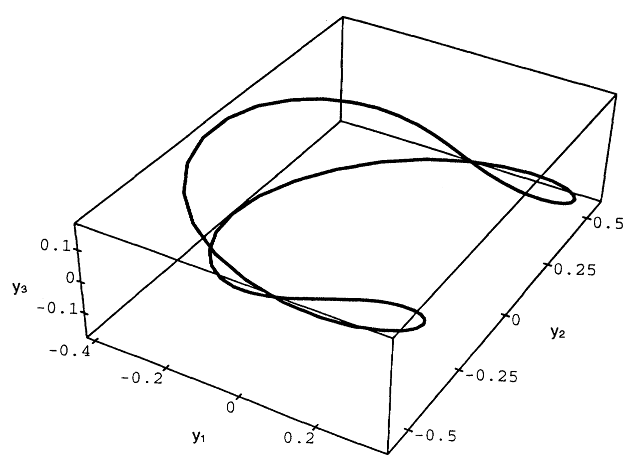

FIG. 9. Phase portrait of a resonant periodic orbit in the chaotic domain III at $F_1 = 0.3$, $A_5 = 0.3$, and $\omega = 0.23$ oscillations per day.

Nevertheless, it has to be examined how well the behavior of the low-order model is reproduced by higher-dimensional truncations of the original infinite-dimensional problem. Some investigations indicate that the behavior of the Lorenz system does not always occur in the same way in higher truncations [22].

In our case of the periodically forced system (1) there is the study of Curry [17] where he investigates the interaction of a set of 14 Fourier modes obtained from the Navier-Stokes equation in the Boussinesq approximation (the Lorenz system can be derived in a similar way from this equation). In his simulation he uses a fixed Prandtl number $\sigma = 2.5$ near to our choice of $a = \sigma = 3$ and the ratio of Rayleigh number to critical Rayleigh number $r = R/R_c$ as a bifurcation parameter. Curry studies a parameter range where attracting invariant tori exist. Since—as we have shown—such objects cannot occur in the Lorenz system, we might expect to see this difference frequently when comparing the Lorenz system and higher-dimensional truncations. Nevertheless Curry claims that these tori play the role of the stationary points in the ordinary Lorenz system (2). Periodic forcing is introduced in a similar way as in our study by periodic modulation of the Rayleigh number, which is proportional to our forcing function F. His observations correspond to ours: On the one hand the system responds periodically to the forcing, on the other hand certain frequencies produce a chaotic response.

Curry's investigation was motivated by Gollub and Benson's experimental study of Rayleigh-Bénard convection [15]. They used a small rectangular fluid cell containing water at 70 °C where the Prandtl number is 2.5. The vertical temperature difference and, accordingly, the Rayleigh number could be modulated. Their system showed qualitatively similar behavior to Curry's 14-mode system although Curry states that his system is not adequate to describe low-Prandtl-number fluids. Obviously the relation of the Lorenz and Curry equations to the results of Gollub and Benson remains unclear, but they show that the qualitative behavior produced by these simple systems can also be observed in real world experiments.

The question of the relationship of the simple models (1) and (2) to the climate system with its variety of interacting cycles is difficult to answer. It is still unknown which of the atmospheric processes (including the convective processes that are modeled in our system) are essential for modeling climate. Even the question how chaotic behavior in the atmosphere is related to chaos in a finite system of ordinary differential equations is still unresolved.

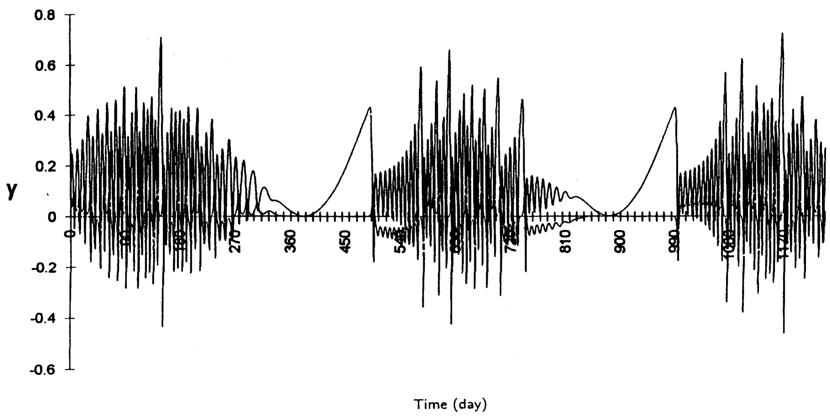

FIG. 10. Periodic change between chaotic and stable behavior at $F_1 = 0.1$, $A_5 = 0.1$, and $\omega = 0.002$ oscillations per day.

A hint in this direction was given by Lorenz [6]. He has shown that the attractor of the quasigeostrophic model (1) is qualitatively like that of the corresponding nine-dimensional system without geostrophic approximation, provided that the intensity of the forcing is not too strong. This confirms that, since a large part of atmospheric processes occurs in geostrophic equilibrium, our model enables at least the representation of a reasonable class of phenomena, but it is obvious that further investigations are necessary before any reliable conclusions concerning climate can be drawn.

VI. CONCLUSION

The effects of introducing periodic forcing into the quasigeostrophic model (1) can be summarized in four points: (1) A global analysis shows that the periodically forced system stays dissipative and bounded. The kinds of attracting sets in the quasigeostrophic model (1) is restricted. Stable three-dimensional manifolds with nonzero volume (including invariant tori) do not exist under its flow, neither do stable equilibrium solutions. The only observable attracting sets are periodic orbits and strange attractors. (2) For forcing frequencies sufficiently far from the eigenfrequency stationary points are replaced by oscillations around them, while chaotic behavior remains essentially unchanged. Periodic forcing is able to stabilize periodic orbits in the chaotic region. (3) Near the eigenfrequency we observe resonance phenomena: (a) It is possible to produce a chaotic response in the stable region. (b) Chaos can be suppressed, if the forcing frequency is appropriately adjusted. (c) New complicated subharmonic orbits are created. (4) If the forcing wanders between different parameter domains we find overlapping effects such as mixed orbits and shifting of domain boundaries.

Some of these effects have been observed in higher-dimensional models and experiments [15,17], but the relation to the climate system remains uncertain. Lorenz concluded that, if the real atmospheric equations behaved like his model, long-range forecasting of specific weather conditions (which also includes climatic phenomena) would be impossible [23]. He explicitly removed annual and diurnal variations in his model while our study shows that periodic variations in the thermal forcing can-

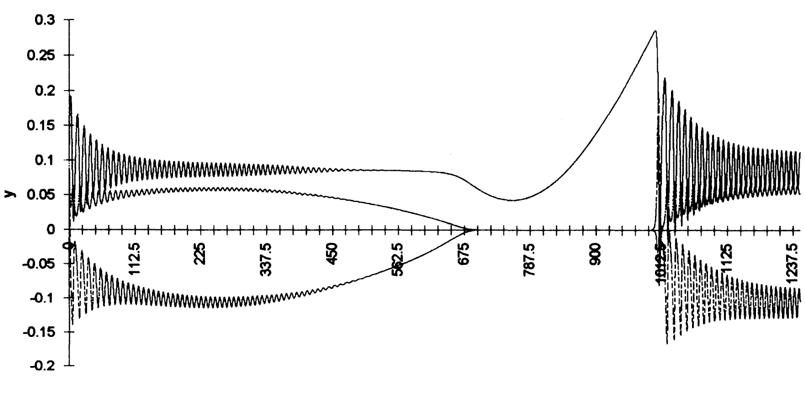

FIG. 11. Periodic change between the quasistatic orbits of domains I and II at $F_1 = 0.055$, $A_5 = 0.047$, and $\omega = 0.001$ oscillations per day.

not only have a considerable effect, but in some cases are even able to suppress chaotic behavior. Certainly his conclusion might still be valid, for chaos in our system is only supressed for certain parameter choices, but still our study implies that periodic cycles in the climate system might play too important a role to be neglected.

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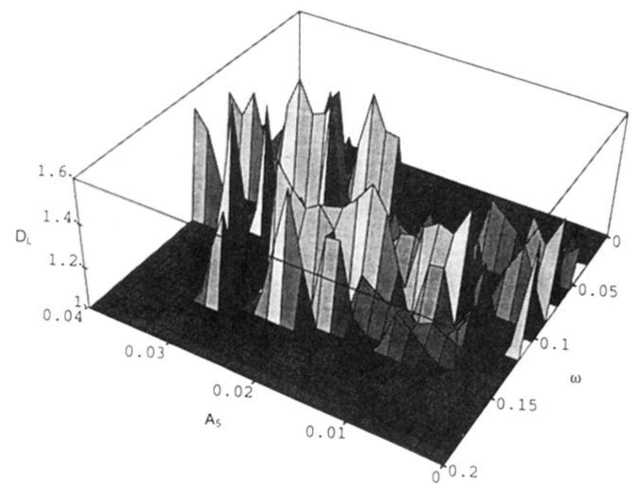

FIG. 1. Lyapunov dimension D_L for $F_1 = 0.04$ (domain II), $A_5 \in [0, 0.04]$ and $\omega \in [0, 0.2]$ (oscillations per day).

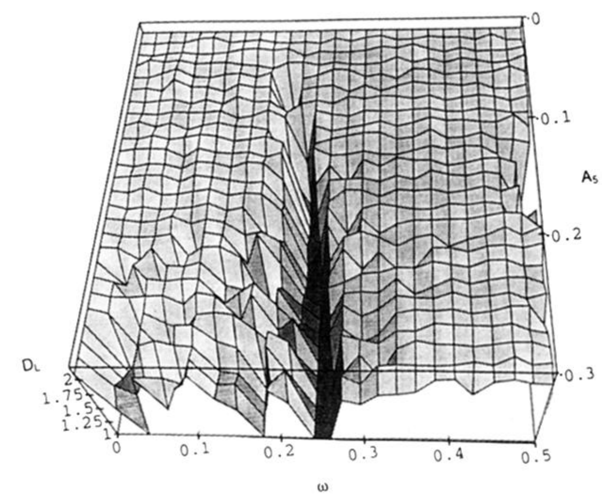

FIG. 2. Lyapunov dimension D_L for $F_1 = 0.3$ (domain III), $A_5 \in [0, 0.3]$ and $\omega \in [0, 0.5]$ (oscillations per day).